Scale-free boundary control of multiple aggregates in large-scale networks

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Abstract: This paper deals with the problem of controlling aggregated linear outputs of a large-scale network to a constant reference value. A multi-output feedback controller is designed such that no information about state vector or system matrices is needed. To provide the conditions for the controller to be stable irrespectively of the gains, the passivity formalism is used and the system is shown to be Strictly Positive Real (SPR). This controller can be used to regulate average states of regions of a large-scale network with control applied to nodes that lie at the boundaries of the regions.

Keywords: Large Scale Systems, Control of Networks, Passivity

1. INTRODUCTION

Control of large-scale network systems is a problem which attracted a lot of attention in a modern control theory society due to its applications to many systems of interest such as urban traffic networks, power networks, social interactions, robot swarms and many more.

In tasks where the number of nodes can reach millions, calculating traditional control algorithms can be too expensive. Moreover, controlling all the states of the system can require a huge amount of energy, which grows exponentially with the size of the system (Yan et al. (2012); Liu et al. (2011)). Thus, in some cases it can be preferrable to control some aggregated characteristics of the entire network rather than all individual states. For example, the output controllability of large-scale networks was studied by Klickstein et al. (2017); Casadei et al. (2018). It was pointed out that the energy required to control aggregated characteristics instead of all network states is much less.

Nikitin et al. (2019, 2020) performed a stabilization of an average state (or a general scalar output) for a large-scale network, assuming that it is stable and positive (that is, the system matrix has positive elements outside the main diagonal). However, in many practical applications it is crucial to control several outputs, for example average states of different parts of the network.

In our work we suppose that neither the measurement of all states, nor the control of all states is a necessary assumption of the network system. It is assumed that *the only values that are measured and regulated are the values of the system outputs*. Moreover, *the system model is not used in the controller*. Thus the equilibrium of internal states is never computed explicitly, and the controller directly utilizes only system outputs and reference point.

A particular example of such setup is a scale-free network, where the goal is to control the averaged state of the hubs and the control is applied to the boundaries of the hubs. Scale-free control of large-scale networks lies in the direction of the research project ERC Scale-FreeBack. Output controllability of a scale-free network is studied in Casadei et al. (2018), the dual problem of reconstruction of average states of several clusters is solved in Niazi et al. (2019), and the reduction of any network to a scale-free is presented in Martin et al. (2019).

The main contribution of our work is a sufficient condition on the system matrices that guarantees stability of any positive integral controller for controlling the multi-input multi-output (MIMO) system to a constant reference point without knowledge of the system matrices.

From the passivity analysis in the classical control theory it is known that the feedback interconnection between a linear operator with an integral controller is stable irrespectively of gain (has an infinite gain margin) if the linear operator is strictly positive real (SPR), see Sepulchre et al. (2012). From this point of view our work provides a new sufficient condition for the network system to be SPR.

2. PROBLEM FORMULATION

We start the problem formulation with an example: assume the system we need to control is the network given by the graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is the set of vertices and $\mathscr{E}$ is the set of edges. The number of vertices $|\mathscr{V}|$ is denoted by n. On each node $v_i \in \mathscr{V}$ the state x_i is defined. Each edge $e \in \mathscr{E}$, where $e = \{v_i, v_j\}$, corresponds to the interaction between nodes v_i and v_j. Matrix $A \in \mathbb{R}^{n\times n}$ represents interaction ratio. Most of real-world networks are internally stable, so we further assume A being stable.

The evolution of the states x and the network output y is given by the following linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{1}$$

where $B \in \mathbb{R}^{n\times k}$ describes how control enters the system, and $C \in \mathbb{R}^{m\times n}$ defines aggregated outputs. For example, if one wants to control the average state, one can choose $C = \frac{1}{n}\mathbf{1}^T$.

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In general, our control goal is to stabilize outputs y over the whole network to some desired constant states y_d without the explicit knowledge of system matrices. It is assumed that the number of states is too large that it is impossible to use full-state feedback or to use matrix A explicitly. Thus, the goal is to find the control law $u = u(y)$ for the system (1) such that

$$\lim_{t\to\infty} y(t) = y_d. \tag{2}$$

Section 3 is devoted to the problem of controlling the single-output system, while Section 4 shows the solution for the problem in case of multi-output systems.

Notation Along this work several types of inequalities are used. If $x \in \mathbb{R}^n$, then $x > 0$ means $x_i \geqslant 0\ \forall i \in \{1..n\}$ and there exists $j \in \{1..n\} : x_j > 0$. If $P \in \mathbb{R}^{n\times n}$, then $P \succ 0$ means P being symmetric positive-definite.

Matrix I denotes the identity matrix of an appropriate size, while column vector of ones is denoted as $\mathbf{1}$.

3. CONTROL OF SINGLE OUTPUT

First, assume that $m = 1$ and thus that there is only one output to be stabilized. In Nikitin et al. (2019, 2020) it was shown that the simplest possible controller able to achieve (2) is the integral controller of the form

$$\dot{u} = \kappa\gamma(y_d - y), \tag{3}$$

where $\gamma \in \mathbb{R}^k$ is the vector of gains, defining relative control force applied to different actuated nodes, and κ is the overall gain. Note that having zero initial conditions for the controller, one can rewrite (3) as

$$\begin{cases} \dot{\alpha} = \kappa(y_d - y), \\ u = \gamma\alpha, \end{cases} \tag{4}$$

and thus come up with the closed-loop system as in Fig. 1.

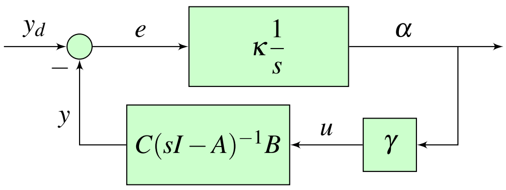


Figure 1. Feedback interconnection of passive systems in SISO case

Here $e = y_d - y$, the closed-loop system input is defined as y_d and the system output is α. With such control decomposition the system (1) becomes SISO with respect to input variable α.

It is known that for L_2 stability of the system with feedback interconnection it is sufficient that the transfer function of one of the blocks is *Positive Real* (PR, which is equivalent to passivity) and another is *Strictly Positive Real* (SPR) (Sepulchre et al. (2012); Kottenstette et al. (2014)).

Define $H_1(s) = \kappa/s$ and $H_2(s) = C(sI - A)^{-1}B\gamma$. Passivity of an integral controller $H_1(s)$ is obvious, so to have stability $H_2(s)$ should be SPR.

Assume that A is a Metzler matrix (its off-diagonal elements are non-negative), which means all edges have positive weights. Such choice of system matrix brings the system (1) to the class of positive systems. Then, in (Nikitin et al. (2020), Lemmas 1, 2 and 3) the following result was proven:

Theorem 1. The scalar transfer function $C(sI - A)^{-1}B\gamma$ is SPR if A is Metzler stable, $CA^2 > 0$ and $CA^2B\gamma > 0$.

It remains to prove that $y \to y_d$ if the closed-loop system is L_2 stable. This is obvious if one recalls that an output of a stable system with constant input converges to a constant value, thus for any constant y_d there exists α^* such that $\alpha \to \alpha^*$. But convergence of an output of an integral controller means that its input converges to zero, which reads as $e \to 0$, which is exactly $y \to y_d$.

4. CONTROL OF MULTIPLE OUTPUTS

Assume the system (1) has special structure, corresponding to the network controlled from the boundaries. Namely, let the state vector be divided into two parts, $x^T = (x_1^T, x_2^T)$. The states $x_1 \in \mathbb{R}^k$ correspond to the boundary nodes, which can be directly controlled, and the states $x_2 \in \mathbb{R}^{n-k}$ are inner nodes, thus no control is applied to them. Assume further that the subnetwork corresponding to the inner nodes is undirected, while the interconnection between inner and boundary nodes exists only in the direction from boundaries to inner nodes, thus there is no influence from inner nodes to boundaries. Schematically this structure is depicted in Fig. 2.

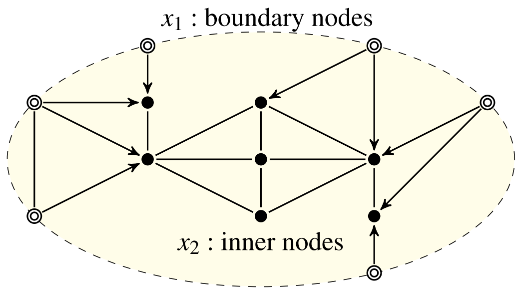


Figure 2. Network with boundary and inner nodes separation

In contrast to the previous section, the goal is to control multiple outputs to their desired values. Note that using the direct control of boundary nodes, the number of outputs m should be the same as the number of inputs k in order to use the passivity formalism. The system model is then:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + u, \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2, \\ \ y = C_1x_1 + C_2x_2, \end{cases} \tag{5}$$

with an additional assumptions that $A_{22} = A_{22}^T \in \mathbb{R}^{(n-k)\times(n-k)}$ is a symmetric negative-definite matrix, corresponding to the undirected stable subnetwork, and that A_{21} is of full rank, meaning that the boundary nodes act independently.

Define the integral controller

$$\dot{u} = \Gamma(y_d - y), \tag{6}$$

where $\Gamma \in \mathbb{R}^{k\times k}$ is a symmetric positive-definite matrix. Then the closed-loop system has the structure as in Fig. 3.

As in the previous section, the accomplishment of (2) follows from $H_1(s) = \Gamma/s$ being PR and $H_2(s) = C(sI - A)^{-1}B$ being SPR. If Γ is positive-definite, $H_1(s)$ is PR. Now we present the theorem which is the main result of the paper:

Theorem 2. If the matrix C_1 is symmetric positive-definite and the matrix inequality (7) is satisfied

$$4H + \delta K - JK^{-1}J \succ 0, \tag{7}$$

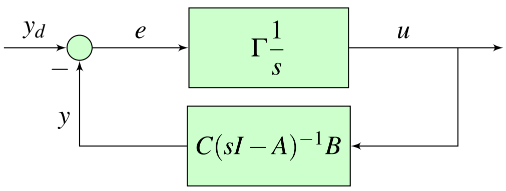


Figure 3. Feedback interconnection of passive systems in MIMO case

with matrices H, J, K and a positive scalar δ defined as follows:

$$
\begin{aligned}
H &= C_2A_{22}C_2^T + C_2C_2^TA_{11} + A_{11}^TC_2C_2^T + A_{11}^TC_2A_{22}^{-1}C_2^TA_{11},\\
J &= 2C_1A_{11} + 2A_{11}^TC_1 + C_2A_{21} + A_{21}^TC_2^T\\
&\quad - A_{21}^TA_{22}^{-1}C_2^TA_{11} - A_{11}^TC_2A_{22}^{-1}A_{21},\\
K &= A_{21}^TA_{22}^{-1}A_{21},
\end{aligned}
\tag{8}
$$

$$
\delta = \begin{cases} \frac{1}{4}\left(\lambda_{max}(JK^{-1}) - \lambda_{min}(JK^{-1})\right)^2, \\ \qquad \text{when} \quad 4\lambda_{max}(C_2^TC_1^{-1}C_2) \leqslant \lambda_{min}(JK^{-1}) + \lambda_{max}(JK^{-1}) \\ 4\left(\lambda_{max}(C_2^TC_1^{-1}C_2) - \frac{1}{2}\lambda_{min}(JK^{-1})\right)^2 \\ \qquad \text{when} \quad 4\lambda_{max}(C_2^TC_1^{-1}C_2) > \lambda_{min}(JK^{-1}) + \lambda_{max}(JK^{-1}) \end{cases}
\tag{9}
$$

then the transfer function of the system (5) is Strictly Positive Real (SPR)

Proof. A known result (Narendra, K.S., Taylor, J.H. (1973)) from passivity theory says that the stable system is SPR iff there exists $P = P^T \succ 0$ such that

$$
\begin{cases} A^TP + PA \prec 0, \\ PB = C^T. \end{cases}
\tag{10}
$$

It was also shown (Tao G, Ioannou P. (1990), Theorem 3.4) that if such matrix P exists, then

$$
\begin{cases} CB = (CB)^T \succ 0, \\ CAB + (CAB)^T \prec 0. \end{cases}
\tag{11}
$$

It is easy to show that the conditions (11) are also sufficient if matrices B and C are square and non-singular. Indeed, take $P = C^TB^{-1}$. Then $B^TPB = (CB)^T = CB = B^TP^TB \succ 0$, from which it is clear that $P = P^T \succ 0$. Then, $CAB + (CAB)^T \prec 0$ implies $B^T(PA + A^TP)B \prec 0$, which is possible only if $A^TP + PA \prec 0$.

In our case only $k < n$ nodes are controlled, thus the control matrix is not square. But we can assume that there exist also $n-k$ "virtual" controls, acting on the inner nodes, such that the modified control matrix is square. Moreover, by the structure of the system real controls form the identity matrix of rank k, thus the reasonable choice for virtual controls is the identity matrix of rank $n-k$.

The observation matrix C should also be augmented, and from the condition $CB = (CB)^T \succ 0$ with identity controls it follows that $C = C^T \succ 0$. Therefore we can define all the matrices,

$$
C = \begin{pmatrix} C_1 & C_2 \\ C_2^T & D \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = I,
$$

where $C_1 = C_1^T \succ 0$ and $D = D^T \in \mathbb{R}^{(n-k)\times(n-k)}$ is some positive-definite matrix which corresponds to the "virtual" observations.

The main reason to augment the system with virtual controls and observations is that once SPR-ness of the augmented system is proven, it immediately implies the that the transfer function of the original system is also SPR. If there exists P such that (10) holds for the augmented system, the same P can be used to prove (10) for the original system. Indeed, the first condition $A^TP + PA \prec 0$ is the same for both systems, and $PB = C^T$ can be decomposed into $PB_i = C_i^T$, where B_i is the i-th column of the control matrix and C_i is the i-th row of the observation matrix, thus for every subset of controls and observations the equality holds.

Now we define a matrix G representing the second condition (11) for the augmented system:

$$
\begin{aligned}
G &= CAB + (CAB)^T = CA + A^TC = \\
&\begin{pmatrix} C_1A_{11} + A_{11}^TC_1 + C_2A_{21} + A_{21}^TC_2^T & C_2A_{22} + A_{11}^TC_2 + A_{21}^TD \\ A_{22}C_2^T + C_2^TA_{11} + DA_{21} & A_{22}D + DA_{22} \end{pmatrix}
\end{aligned}
\tag{12}
$$

The main question is if there exists such matrix $D = D^T \succ 0$ that $C \succ 0$ and $G \prec 0$. In general this question is very hard to answer, but we can restrict our attention to a special class of matrices D such that the sufficient conditions on C and A can be obtained. Namely, let us choose $D = \alpha I$ for some positive scalar α. Then if we can find α such that $C \succ 0$ and $G \prec 0$, the system is SPR. With the new variable α the matrix G becomes

$$
G = \begin{pmatrix} C_1A_{11} + A_{11}^TC_1 + C_2A_{21} + A_{21}^TC_2^T & C_2A_{22} + A_{11}^TC_2 + \alpha A_{21}^T \\ A_{22}C_2^T + C_2^TA_{11} + \alpha A_{21} & 2\alpha A_{22} \end{pmatrix}.
\tag{13}
$$

By Schur's Complement, $C \succ 0$ leads to the condition $\alpha I - C_2^TC_1^{-1}C_2 \succ 0$, thus α should satisfy

$$
\alpha > \lambda_{max}(C_2^TC_1^{-1}C_2)
\tag{14}
$$

And applying the Schur's Complement to the matrix G (and recalling that A_{22} is negative-definite) we see that $G \prec 0$ is equivalent to

$$
\begin{aligned}
&2\alpha(C_1A_{11} + A_{11}^TC_1 + C_2A_{21} + A_{21}^TC_2^T) - \\
&(C_2A_{22} + A_{11}^TC_2 + \alpha A_{21}^T)A_{22}^{-1}(A_{22}C_2^T + C_2^TA_{11} + \alpha A_{21}) \prec 0
\end{aligned}
\tag{15}
$$

Using the definitions (8) and removing brackets we get

$$
\alpha^2K - \alpha J + H \succ 0,
\tag{16}
$$

where $K \prec 0$ because $A_{22} \prec 0$ and A_{21} is of full rank, and J and H are in general sign-indefinite.

Define $L = (-K)^{-1/2}$, where square root is chosen such that L is positive definite. Multiplying (16) from both sides by L, we obtain

$$
\alpha^2I + \alpha LJL - LHL \prec 0,
\tag{17}
$$

which can be rearranged as

$$
\left(\alpha I + \frac{1}{2}LJL\right)^2 \prec L\left(H - \frac{1}{4}JK^{-1}J\right)L
\tag{18}
$$

If it had been a scalar quadratic equation, it would be possible to make the left-hand side zero by choosing appropriate α. In our case it is not possible, but we can find an upper bound on this term. Namely,

$$
\left(\alpha I + \frac{1}{2}LJL\right)^2 \prec \lambda_{max}\left[\left(\alpha I + \frac{1}{2}LJL\right)^2\right]I,
\tag{19}
$$

where

$$\begin{aligned}&\lambda_{max}\left[\left(\alpha I+\frac{1}{2}LJL\right)^2\right]=\\&\max\left\{\lambda_{max}\left(\alpha I+\frac{1}{2}LJL\right),-\lambda_{min}\left(\alpha I+\frac{1}{2}LJL\right)\right\}^2=\\&\max\left\{\alpha+\frac{1}{2}\lambda_{max}(LJL),-\alpha-\frac{1}{2}\lambda_{min}(LJL)\right\}^2=\\&\max\left\{\alpha-\frac{1}{2}\lambda_{min}(JK^{-1}),-\alpha+\frac{1}{2}\lambda_{max}(JK^{-1})\right\}^2.\end{aligned}\tag{20}$$

The minimal value is achieved when two arguments of maximum are equal. Define $\alpha_1^*=\frac{1}{4}\left(\lambda_{min}(JK^{-1})+\lambda_{max}(JK^{-1})\right)$. Then the bound is

$$\begin{aligned}\delta_1:=\lambda_{max}&\left[\left(\alpha_1^* I+\frac{1}{2}LJL\right)^2\right]=\\&=\frac{1}{16}\left(\lambda_{max}(JK^{-1})-\lambda_{min}(JK^{-1})\right)^2.\end{aligned}\tag{21}$$

This value is optimal, but it assumes that α_1^* can be used, which is possible only if (14) is satisfied. Denote $\alpha_2^*=\lambda_{max}(C_2^T C_1^{-1} C_2)$. If $\alpha_2^*>\alpha_1^*$, then the bound is

$$\begin{aligned}\delta_2:=\lambda_{max}&\left[\left(\alpha_2^* I+\frac{1}{2}LJL\right)^2\right]=\\&=\left(\alpha_2^*-\frac{1}{2}\lambda_{min}(JK^{-1})\right)^2=\\&=\left(\lambda_{max}(C_2^T C_1^{-1} C_2)-\frac{1}{2}\lambda_{min}(JK^{-1})\right)^2.\end{aligned}\tag{22}$$

Finally, defining $\delta:=4\delta_1$ if $\alpha_2^*\leqslant\alpha_1^*$ and $\delta:=4\delta_2$ otherwise, we can rewrite the quadratic equation (18) as

$$\left(\alpha I+\frac{1}{2}LJL\right)^2\prec\frac{1}{4}\delta I\prec L\left(H-\frac{1}{4}JK^{-1}J\right)L,\tag{23}$$

and, multiplying both sides by L^{-1}, get a sufficient condition

$$-\delta K\prec 4H-JK^{-1}J,\tag{24}$$

which is exactly the condition (7) that we aimed to obtain. □

Remark. The result of the theorem is only a sufficient condition for the transfer function to be SPR. The augmentation of the system with virtual controls and observations is still an equivalence operation, as one can always choose such additional columns for B and C^T that $PB=C^T$ holds. The equivalence is lost when the matrix D is substituted with αI. For the future work it would be possible to add an additional degree of freedom to this procedure by considering an augmentation of matrix B in the form

$$B=\begin{pmatrix}I&0\\0&\beta I\end{pmatrix},$$

thus obtaining a system of two quadratic matrix inequalities on α and β. Possibly tighter sufficient conditions could be recovered as a result.

5. EXAMPLE: UNDIRECTED LINE NETWORK

Inequality (7) is a straightforward condition to check, given the system, but it is hard to interpret in general. However from the definitions of matrices H, J and K it is clear that the condition is easier to satisfy for bigger C_1, for C_2 closer to zero and for bigger K and $-K^{-1}$. The latter happens when A_{22} is strongly negative, for example when the inner nodes have strong negative self-loops.

It can be shown on the example of an undirected line network, controlled from two sides, with the aim to stabilize two halves of the network to different average values. The network for $n=8$ nodes is depicted in Fig. 4.

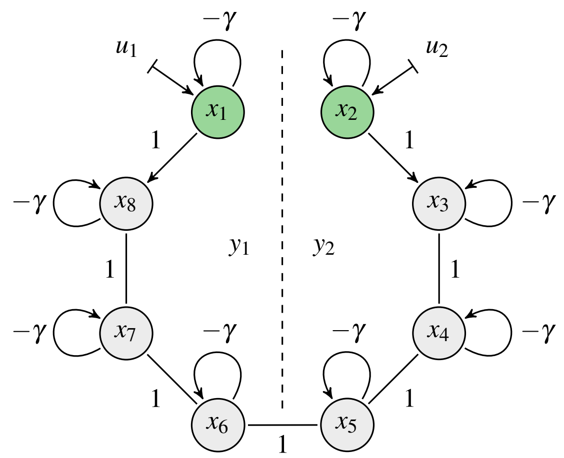


Figure 4. Undirected line network, controlled from two sides, with $n=8$ nodes. Boundary nodes are in green. Network is splitted into two parts by dahsed line, denoting two separate outputs y_1 and y_2.

Dynamics of this network are given by the matrices

$$A=\left(\begin{array}{cc|ccccc}-\gamma&0&0&0&\cdots&0&0\\0&-\gamma&0&0&\cdots&0&0\\\hline 0&1&-2-\gamma&1&\cdots&0&0\\0&0&1&-2-\gamma&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&0&\cdots&-2-\gamma&1\\1&0&0&0&\cdots&1&-2-\gamma\end{array}\right),\tag{25}$$

$$C=\left(\begin{array}{cc|ccc|ccc}1&0&0&\cdots&0&\alpha&\cdots&\alpha\\0&1&\alpha&\cdots&\alpha&0&\cdots&0\end{array}\right),\qquad B=\begin{pmatrix}I\\0\end{pmatrix}.\tag{26}$$

This system can be either SPR or not for given n, depending on the parameters α and γ. Intuitively, lower is α, smaller is the influence of C_2, and easier is to obtain passivity. In the same way if γ is large enough, A_{22} is strongly negative and thus system is passive. Indeed, in Fig. 5 the smallest eigenvalue λ_{min} of the matrix $4H+\delta K-JK^{-1}J$ is depicted for $n=100$ depending on both α and γ. The condition (7) is satisfied when $\lambda_{min}>0$.

The dependence of the condition (7) on the size of the network is presented in Fig. 6, where the same $\lambda_{min}(4H+\delta K-JK^{-1}J)$ is depicted for $\alpha=0.6$ for various γ and n. It is clear that for a longer line network to be passive the negative self-loops should be stronger.

Now we apply an integral controller to this system, with a goal to stabilize the outputs to the desired values $y_{d1}=5$ and $y_{d2}=10$. The controller has the form

$$\dot{u}=\begin{pmatrix}\kappa&0\\0&\kappa\end{pmatrix}\begin{pmatrix}y_{d1}-y_1\\y_{d2}-y_2\end{pmatrix}.\tag{27}$$

Simulation results for $n=100$ and $\kappa=50$ are shown in Fig. 7. Indeed, for fixed value of $\alpha=0.6$, small value of $\gamma=0.5$ leads

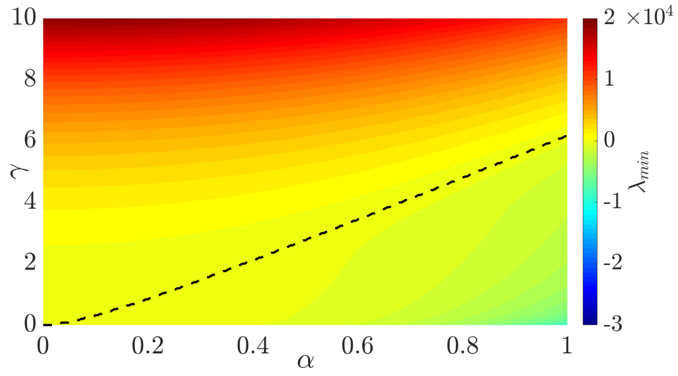


Figure 5. Smallest eigenvalue $\lambda_{min}(4H + \delta K - JK^{-1}J)$, depending on α and γ for $n = 100$. Dashed line denotes zero level. All points above dashed line satisfy (7).

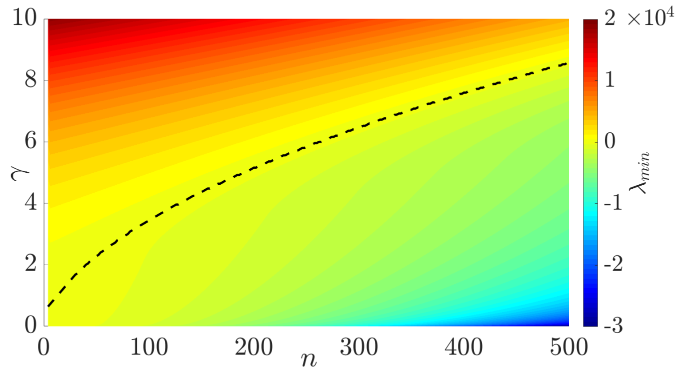
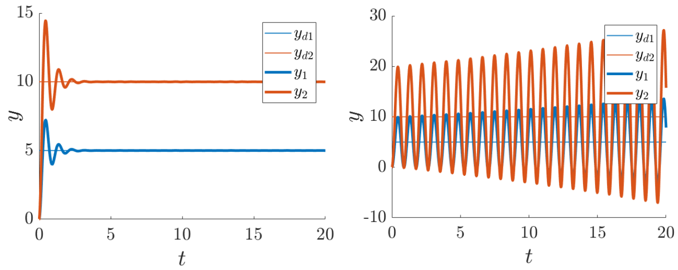


Figure 6. Smallest eigenvalue $\lambda_{min}(4H + \delta K - JK^{-1}J)$, depending on n and γ for $\alpha = 0.6$. Dashed line denotes zero level. All points above dashed line satisfy (7).

to the unstable behaviour of the closed-loop system, while $\gamma = 4$ is stable.



(a) (b)

Figure 7. Multi-output control of the undirected line network, $n = 100$, $\kappa = 50$, $\alpha = 0.6$. Inset (a): $\gamma = 4$, the closed-loop system is stable. Inset (b): $\gamma = 0.5$, the closed-loop system is unstable.

6. CONCLUSION

In this paper we considered problem of multi-output control of a large scale network system. We applied a simple integral controller to the system and showed that the output stabilization is achieved when the transfer function of the system is SPR.

We derived a sufficient condition on the system matrices for the multi-output system to be SPR. If the system satisfies this condition, there is no need to adjust the parameters of the controller, to have knowledge of the state vector or the values of the elements of the system matrix.

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